Spectrum of geometric phases in a driven impact oscillator

2005 J. Phys. A: Math. Gen. 38 485

(http://iopscience.iop.org/0305-4470/38/2/013)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 485–493 doi:10.1088/0305-4470/38/2/013

Spectrum of geometric phases in a driven impact oscillator

Indubala I Satija[1] **and Radha Balakrishnan**[2]

[1] Department of Physics, George Mason University, Fairfax, VA 22030, USA
[2] The Institute of Mathematical Sciences, Chennai 600 113, India

Received 27 August 2004, in final form 20 October 2004
Published 15 December 2004
Online at stacks.iop.org/JPhysA/38/485

Abstract
We study the geometric phases underlying the time evolution of the quantum wavefunction of a driven nonlinear oscillator which exhibits periodic, quasiperiodic as well as chaotic dynamics. In the asymptotic limit, irrespective of the classical dynamics, the geometric phases are found to increase linearly with time. Interestingly, the fingerprints of the classical motion are present in the bounded fluctuations that are superimposed on the monotonically growing phases, as well as in the difference between the geometric phases of two neighbouring quantum states.

PACS numbers: 03.65.Vf, 05.45.Mt, 05.45.Ac

1. Introduction

The geometric phase has emerged as a new concept at the forefront of physics providing a geometric framework for understanding various fundamental problems (Shapere and Wilczek 1989). This activity began after a discovery by Berry (1984) which showed the existence of a phase of purely geometric origin, which is acquired by an eigenfunction in an adiabatic time evolution. This deep and elegant concept has been extended to non-adiabatic cases (Aharonov and Anandan 1987) and also to non-cyclic circuits (Samuel and Bhandari 1988) and hence applies to essentially any type of quantum evolution. Using a kinematic approach (Mukunda and Simon 1993), the geometric phase has been shown to be a gauge-invariant quantity, and is equal to the difference between the total phase and the dynamical phase acquired by the wavefunction. The geometric phase characterizing quantum dynamics is an important topological quantity that is resilient to certain types of errors.

In this paper, we address the question of how a *quantum* geometric phase responds to the changes in the *classical* dynamics, in particular to the transitions from regular to chaotic dynamics. As an illustrative example, we compute the geometric phases associated with the time evolution of the wavefunction of a driven nonlinear oscillator that exhibits periodic, quasiperiodic as well as chaotic dynamics. In the asymptotic limit, these phases increase linearly with time, irrespective of the nature of the corresponding classical dynamics. The key result of our analysis is that the detailed information about the classical dynamics is present in the fluctuations about this monotonic behaviour and also in the difference between the geometric phases associated with two neighbouring quantum states.

2. Classical dynamics of the driven impact oscillator

The Hamiltonian under investigation describes an impact oscillator (Shaw and Holmes 1983, Thompson and Stewart 1986). This is an oscillator that rebounds elastically whenever its displacement x drops to zero, and is driven by an external time-dependent force $F(t)$. For $x > 0$, this system is described by

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega_0^2 x^2 - F(t)x, \tag{1}$$

where the mass of the particle has been set to be unity. The *classical* dynamics of this model has been investigated in a variety of contexts which include reflection of cosmic rays, wave dynamics near shores and the study of various visco-elastic impacts (Thompson and Stewart 1986). Additionally, the dynamics of the impact oscillator has been related to the dual billiard map (Boyland 1994).

The impact oscillator is a piecewise linear system, where the classical analytic solutions $x_c(t)$ can be obtained for $x > 0$. The particular solution for arbitrary F can be written as

$$x_c(t) = (\omega_0)^{-1} \int_0^t [F(t') \sin \omega_0(t - t')]\, \mathrm{d}t', \tag{2}$$

to which any arbitrary solution for the unforced system with $F(t) = 0$ can be added. The dynamics of this nonlinear system is studied numerically by using the analytic solution for $x(t) > 0$ for very small time increments, so as to determine precisely the time and the velocity when x reaches 0. These are then used as the new initial conditions for dynamics. In the presence of damping, the system has been shown to exhibit generic nonlinear effects including a period doubling route to chaos (Thompson and Stewart 1986).

We choose $F(t) = f \cos(\omega t)$, with $\omega = 1$ and $f = 1$. We first explore the classical phase space by varying ω_0 and the initial conditions. The system is found to possess highly complex dynamics, typical of a nonintegrable Hamiltonian system exhibiting stochastic behaviour. Figure 1 shows a mixed phase space portrait illustrating extreme sensitivity to initial conditions. In this paper, we present our results for a fixed $\omega_0 = 1.6$.

3. Quantum dynamics of the impact oscillator

Our choice of the driven impact oscillator is motivated by the fact that in addition to the simplicity underlying the classical analysis of the oscillator, the quantum wavefunctions $\psi(x, t)$ are explicitly known in terms of the classical solution $x_c(t)$ (Rapp 1970, Nogamin 1991):

$$\psi(x, t) = \chi(x', t) \exp \frac{\mathrm{i}}{\hbar} \left[\dot{x}_c(t)x' + \int_0^t L(t')\, \mathrm{d}t' \right]. \tag{3}$$

Here, $x' = [x - x_c(t)]$ and L is the Lagrangian of the driven system. $\chi(x, t)$ denotes the wavefunction of the impact oscillator in the *absence* of driving.

If we take χ to be an eigenstate of the undriven oscillator, $\chi_n(x, t) = u_n(x) \exp -\mathrm{i}(E_n t/\hbar)$, the eigenfunctions u_n can be written in terms of Hermite polynomials as $u_n(x) = \exp -[\omega_0 x^2/(2\hbar)] H_n(x\sqrt{\omega_0/\hbar})$. We would like to point out that the above solution has

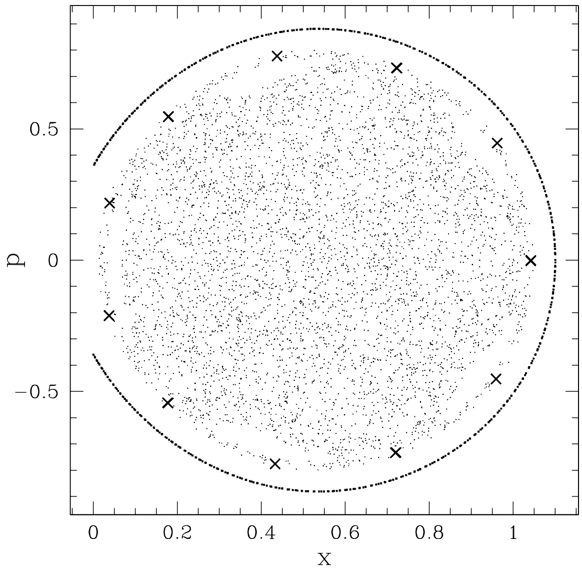

Figure 1. A phase portrait of the driven impact oscillator at discrete times t that are multiples of the driving period $2\pi/\omega$: to show sensitive dependence on the initial conditions, we choose three different initial x values, x_0, with initial momentum $p_0 = 0$. For $x_0 = 1.0416$, there is a period-11 orbit, seen with 11 crosses in the figure. For $x_0 = 1$, the motion is chaotic, and hence appears as randomly scattered dots. With $x_0 = 1.1$, the motion is quasiperiodic, leading to a smooth invariant torus (darker dots) that surrounds the chaotic sea.

been discussed in the literature only in the context of a driven simple harmonic oscillator, whose classical dynamics is linear. On the other hand, the dynamics of the impact oscillator is nonlinear, since it is a simple harmonic oscillator with an additional boundary condition such that the dynamics is confined to $x > 0$. Therefore, the wavefunction of the simple harmonic oscillator that vanishes at $x = 0$ describes the impact oscillator. (This is analogous to the example of a particle in a box, whose quantum wavefunction is the free particle wavefunction that vanishes at the boundary.) This in turn implies that the eigenfunctions of the impact oscillator are Hermite polynomials, but with n restricted to odd integers as they vanish at $x = 0$ as required. However, it should be emphasized that due to the explicit dependence on $x_c(t)$, the quantum wavefunction of the driven impact oscillator is very different from that of the driven simple harmonic oscillator. The centre of the wave packet, given by $x_c(t)$, will exhibit periodic, quasiperiodic as well as chaotic dynamics.

We would like to point out that wavefunction given by equation (3) is not an eigenstate of the Hamiltonian (1). The adiabatic wavefunction of the driven oscillator is given by $\chi(x - (\omega_0)^{-2}F(t))$ with eigenvalues $E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0 - \frac{1}{2}\omega_0^{-2}F^2(t)$. It coincides with u_n provided $x_c = \omega_0^{-2}F(t)$, which is satisfied only when $\ddot{x}_c = 0$. In fact, the quantum wavefunction can be expanded in terms of adiabatic eigenfunctions, or in terms of the eigenstates of the undriven oscillator for computing the transition probabilities (Rapp 1970, Nogamin 1991).

In view of the fact that we have an analytic solution (equation (2)) for the classical system for $x > 0$, and also a closed form solution (equation (3)) for the quantum wavefunction, we can compute the quantum geometric phases with extreme precision, as discussed below.

4. Gauge-invariant geometric phase

We use the kinematic formulation (Mukunda and Simon 1993) to obtain the geometric phase associated with the wavefunction given in equation (3):

$$\Phi(t) = \arg \int_0^\infty \mathrm{d}x[\psi^*(x,0)\psi(x,t) - \mathrm{Im} \int_0^t \psi^*(x,t')\partial_{t'}\psi(x,t')\,\mathrm{d}t']. \tag{4}$$

Here, the first and second terms represent, respectively, the total phase and the dynamical phase accumulated by the wavefunction in time t. There exists a spectrum of phases characterized by an odd integer n associated with the wavefunction (3), with $\chi = \chi_n$ being the eigenstate of the *undriven* impact oscillator. Simple algebraic manipulations show that the geometric phase can be expressed in terms of the expectation value $\langle x \rangle$:

$$\hbar\Phi(t) = -\int_0^t \left[\dot{x}_c^2(t') - \langle x - x_c\rangle \ddot{x}_c(t')\right] \mathrm{d}t' - G(t), \tag{5}$$

where $G(t)$ is given by

$$G(t) = (\dot{x}_c(t)x_c(t) - \dot{x}_c(0)x_c(0)) + \arg\left[\int_0^\infty \mathrm{d}x[(u_n(x'(0))u_n(x'(t))\exp\frac{\mathrm{i}x}{\hbar}(\dot{x}_c(t) - \dot{x}_c(0))\right]. \tag{6}$$

It should be noted that $G(t)$ depends on the initial and end points of the classical motion. Further, it will vanish if we restrict the initial and end points of the motion to be at turning points. In view of this, we will calculate the geometric phases at times where the classical particle is at its turning points.

5. Spectrum of geometric phases

For a driven simple harmonic oscillator, $\langle(x - x_c)\rangle$ vanishes, and therefore, from equation (5), we see that (the modulus of) the geometric phase is equal to twice the kinetic energy of the classical oscillator. This is in agreement with earlier studies (Song 1999). However, for the impact oscillator, $\langle(x - x_c)\rangle$ is non-vanishing and depends upon the quantum number n. We write the geometric phase given in equation (5) as

$$\hbar\Phi = \hbar\phi_0 + \hbar\phi_n, \tag{7}$$

where $\hbar\phi_0$ is the first term in equation (5) and denotes the n-independent contribution to the phase. The second term, $\hbar\phi_n$, may depend on n. We will focus on ϕ_n, the term of quantum mechanical origin. Note also that it is nonzero only for the nonlinear case.

Due to the polynomial character of the Hermite polynomials, the calculation of $\langle(x - x_c)\rangle$ involves computing integrals of the form

$$S_m(x_c) = \int_0^\infty (x - x_c)^{m-1} \exp -\frac{1}{2}(x - x_c)^2\,\mathrm{d}x, \tag{8}$$

where m is an integer. This integral can be evaluated explicitly and the result can be expressed as a summation over parabolic cylinder functions, $D_m(y_c)$. Here, $y_c = \alpha x_c$ with $\alpha = \sqrt{\frac{2\omega_0}{\hbar}}$. It can be shown (Gradshteyn and Ryzhik 1965) that

$$S_m(y_c) = \Gamma(m) \sum_{j=1}^m \frac{(-y_c)^{m-j}}{\Gamma(m+1-j)} D_j(y_c). \tag{9}$$

Therefore, $\langle \delta y \rangle \equiv \langle (y - y_c) \rangle$ can be written in terms of S_m, using the explicit expressions for the Hermite polynomials:

$$\langle \delta y \rangle_n = \langle \delta y \rangle_{(2p-1)} = \frac{\sum_{j=2}^{2p} I_j S_{2j}(y_c)}{\sum_{j=2}^{2p} I_j S_{2j-1}(y_c)}. \tag{10}$$

Here, the coefficients I_j are integers determined by the Hermite polynomials, and $p = 1, 2, 3, \ldots$. For example, for the three lowest states, we obtain

$$\langle \delta y \rangle_1 = \frac{S_4}{S_3}, \tag{11}$$

$$\langle \delta y \rangle_3 = \frac{S_8 - 6S_6 + 9S_4}{S_7 - 6S_5 + 9S_3}, \tag{12}$$

and

$$\langle \delta y \rangle_5 = \frac{S_{12} - 20S_{10} + 130S_8 - 300S_6 + 225S_4}{S_{11} - 20S_9 + 130S_7 - 300S_5 + 225S_3}. \tag{13}$$

In terms of the scaled variable y, the general expression is

$$\phi_n(t) = \frac{1}{2\omega_0} \int_0^t \ddot{y}_c \langle \delta y \rangle_n \, \mathrm{d}t', \tag{14}$$

with n odd. Note that the dependence on $\hbar$ has been absorbed in $y = \alpha x$. The results shown here are for a fixed $\alpha = 2$. The variation with α did not lead to any changes in the behaviour of the phases. Further, as $\hbar \to 0$, asymptotic expansion of the parabolic cylinder function is not well defined near $x \approx 0$. We believe that, in analogy with the harmonic oscillator case, the large n limit may describe the semiclassical limit of the system.

The variation of the geometric phase with time is computed by numerically integrating equation (14). It is instructive to plot the geometric phase per unit time or average phase, which we denote as $\langle \phi_n(t) \rangle = \phi_n(t)/t$, as a function of time t. This is shown in figure 2. Our detailed studies for various parameter values, extending over long time intervals, suggest that as t becomes large, $\phi_n(t)$ varies linearly with time, with small fluctuations superimposed on it. Thus, $\langle \phi_n(t) \rangle$ approaches a constant as $t \to \infty$. Here, we remark that times much longer than those shown in figure 2 were also studied and showed the same behaviour. This appears to be the case for periodic, quasiperiodic as well as chaotic dynamics. These results are counterintuitive and this type of 'convergence' of $\langle \phi_n(t) \rangle$ is particularly surprising for chaotic dynamics. Furthermore, it is rather intriguing that this *asymptotic behaviour* of $\langle \phi_n \rangle$ appears to be insensitive to the underlying classical dynamics, in spite of the fact that the quantum wavefunctions depend explicitly on the classical solution. Note, however, that the *asymptotic values* of $\langle \phi_n \rangle$ do depend on the type of classical dynamics.

It is natural to ask if the behaviour of $\langle \phi_n \rangle$ can be studied analytically, as $n \to \infty$. Using the asymptotic expression for Hermite polynomials for odd-n (Spanier and Oldham 1987), under certain approximations, we can show that as $n \to \infty, \hbar \langle \phi_n \rangle$ approaches the time-averaged kinetic energy of the classical system, and is therefore independent of n. Further, we have verified that for regular classical motion, our value of $\langle \phi_n \rangle$ computed in this large n limit is indeed consistent with the trend suggested by our numerical values for $n = 1, 3, 5$ given in figure 2.

We show below, two different ways in which the quantum geometric phases exhibit some fingerprints of the underlying classical dynamics. Firstly, we study the sensitivity of these phases to initial conditions (figure 3). Secondly, we focus on the small fluctuations by subtracting the linear time dependence from the geometric phases (figure 4).

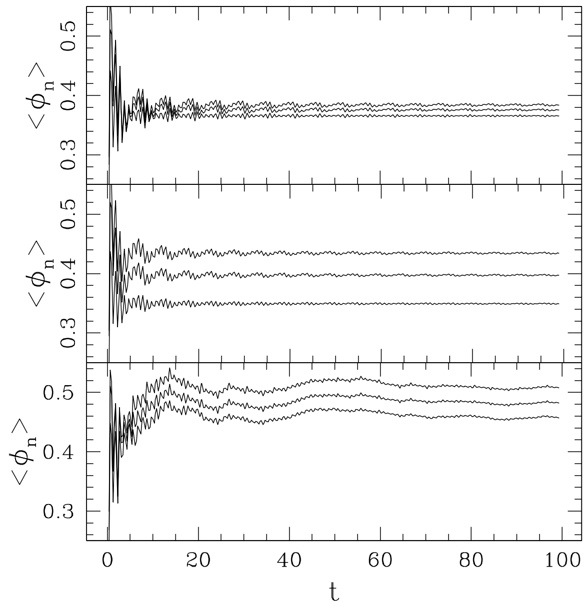

Figure 2. The top, middle and bottom figures show, respectively, the plots of $\langle\phi_n(t)\rangle$ corresponding to the periodic, quasiperiodic and chaotic orbits of the classical system displayed in figure 1. The three curves in each of the figures describe the three quantum states with $n = 1, 3, 5$ (from bottom to top). In every case, asymptotically in time, $\langle\phi_n(t)\rangle$ approaches a constant with very small fluctuations superimposed on it. (The time t is measured in units of $2\pi/\omega$.)

Figure 3 correlates the variation in the classical motion with the corresponding changes in the asymptotic values of the quantum geometric phases $\langle\phi_n\rangle$, as we vary the initial conditions of the oscillator. In the parametric window corresponding to periodic dynamics, the phases remain almost constant independent of the initial condition, in spite of the fact that the period of the motion is changing. In the quasiperiodic regime, they vary smoothly with the initial conditions. In the chaotic regime, they depend somewhat erratically on the initial energy. Briefly, the transition from regular to irregular motion leaves its fingerprints on the spectrum of geometric phases, although remaining somewhat insensitive to the details of the regular motion.

Another way to illustrate the signature of classical motion on the quantum geometric phase is as follows. As suggested by figure 2, asymptotically in time, $\langle\phi_n(t)\rangle$ goes to a constant with small fluctuations superimposed . Thus, we write

$$\phi_n(t) = v_n t + \delta\phi_n(t), \tag{15}$$

where v_n becomes t independent. The $\delta\phi_n(t)$ are bounded oscillatory functions superimposed on the steady growth.

Intriguingly, as shown in figure 4, the details of the classical dynamics, including the period of the classical orbit, are present in $\delta\phi_n$, the fluctuating component of the geometric phase. For periodic dynamics, $\delta\phi_n$ are periodic with the period of the motion. Topological quantum response to chaotic motion is reflected in $\delta\phi_n$ that exhibits chaotic dynamics.

The average phase becoming independent of n for large n, implies that v_n is independent of n for large n. Thus, equation (15) shows that as $n \to \infty$, the difference in the phases between two neighbouring states, denoted by $\mathrm{d}\phi_n$, is just equal to the difference in the small fluctuations that were associated with the phases: $\mathrm{d}\phi_n = (\delta\phi_n - \delta\phi_{n+1}) \to (\phi_n - \phi_{n+1})$. (Again, the trend

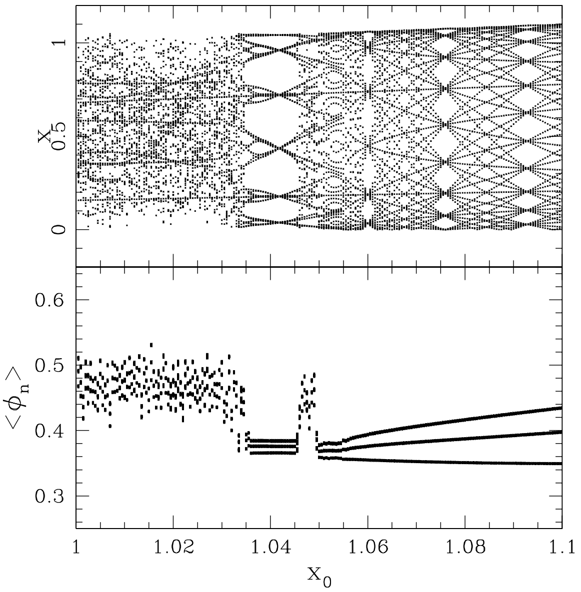

Figure 3. The upper figure shows the *classical* bifurcation diagram, a plot of x at Poincare points, as x_0 (or the initial energy) is varied. Note that the dynamics to the left of the periodic window is mostly stochastic while that to the right is mostly quasiperiodic. The period-6 in x near $x_0 \approx 1.0416$ in fact corresponds to period-11 orbit in $x - p$ as shown in figure 1. The lower figure shows the corresponding quantum topological response to changes in the initial conditions, measured in terms of asymptotic value of $\langle \phi_n \rangle$, with $n = 1, 3, 5$.

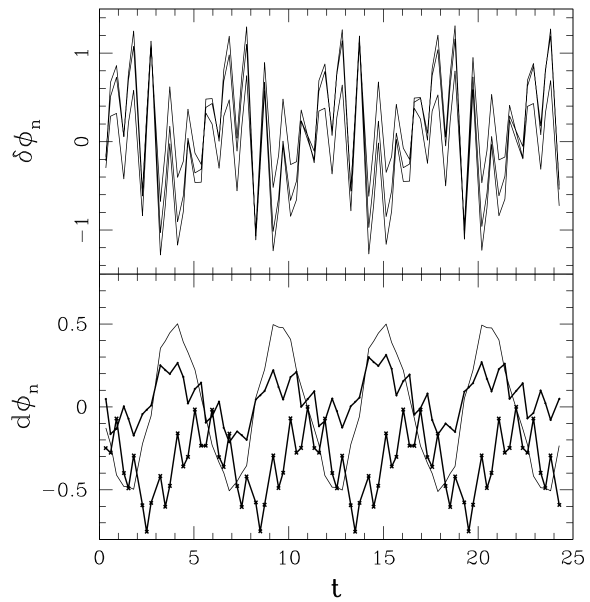

Figure 4. The upper figure shows $\delta\phi_n$ with $n = 1, 3, 5$ for the period-11 trajectory of figure 1. Note that $\delta\phi_n$ also has period-11. The lower figure shows the corresponding $d\phi_n$ with $n = 1$ (light curve) and $n = 3$ (darker curve). The lowest curve (the curve with crosses) shows the classical phase.

for $n = 1, 3, 5$ results for the average phase for the periodic and quasiperiodic cases (figure 2) is consistent with this large n result.)

The fact that the difference in the phases between two neighbouring quantum states retains full information about the corresponding classical dynamics is reminiscent of Berry's relation, $\theta = -\partial_n \phi_n$, where θ is the classical geometric phase or the Hannay angle (Hannay 1985). This relation was established for integrable systems in the semiclassical limit. There have been some attempts to describe the classical limit of the Berry phase for chaotic systems where the effort has been focused on finding a generalization of the *2-form* within a Wigner–Weyl formalism (Robbins and Berry 1992). However, the problem of finding the classical limit of a quantum geometric phase for a chaotic system has remained open.

Our recent preliminary studies suggest that the quantum $\mathrm{d}\phi_n$ may be related to a new type of classical geometric phase, the one that can be associated with the nonplanar phase space trajectories (Balakrishnan and Satija 2003, Satija and Balakrishnan 2004). The discussion of this classical phase is beyond the scope of this paper. However, in figure 4 we compare this classical phase with the difference in the fluctuations and find that they both have the same order of magnitude. This type of correspondence between the classical phase and $\mathrm{d}\phi_n$ is also seen for quasiperiodic as well as for chaotic motions.

6. Summary and conclusions

There have been earlier studies of geometric phases for harmonic oscillators and their generalizations (Song 1999, Seshadri 1999). The work described here is the first study of quantum geometric phases in a driven *nonlinear* oscillator exhibiting chaotic dynamics. We have shown that the geometric phase for this system increases linearly with time, with small fluctuations superimposed, for large times irrespective of the underlying classical dynamics. The nature of the corresponding classical motion is encoded in these fluctuations. The fact that the details of the classical dynamics are mirrored in the difference between the geometric phases of two neighbouring quantum states may be an important result, and its relation to the Berry formula may provide an alternative view for finding the classical limit of quantum phases.

In generalizing our conclusions, one must be cautious as the model under investigation here is related to the harmonic oscillator and therefore certain details such as equation (3) may not have any general validity. However, our key result that the difference in the quantum phases bears fingerprints of classical dynamics may be true in other systems. The importance of a topological description had emerged in an earlier study, in the context of a kicked Harper model with a toroidal geometry, where the spectrum of topological numbers were correlated with the regular to chaotic transition (Leboeuf *et al* 1990). Therefore, the study of topological quantum response may provide a useful framework to address certain open issues in quantum chaos.

In conclusion, Hamiltonians of the form $H(x,t) = H_0 + \lambda x \sin(\omega t)$ are indeed relevant in atom-optics experiments, where H_0 is the time-independent Hamiltonian and the time-dependent term describes the interaction with a single-mode radiation field in the dipole approximation. Optical dipole traps which are close approximations to impact oscillators can be realized in the laboratory, as atoms are repelled from the region where the laser intensity is highest (Davis *et al* 1995). Considerable efforts are underway in using these systems for quantum computation (Brennen *et al* 2003). The usefulness of Berry phase and its generalization has been discussed in constructing fault-tolerant logic gates in nuclear magnetic resonance experiments (Jones *et al* 2000). In addition, the relevance of generalized geometric phases in quantum computing has been the subject of various investigations

(Zanardi and Rasetti 1999, Du *et al* 2002). We hope that our results would stimulate further experimental and theoretical studies of geometric phases.

Acknowledgment

The research of IIS is supported by National Science Foundation grant no. DMR 0072813.

References

Aharanov Y and Anandan J 1987 *Phys. Rev. Lett.* **58** 1593
Balakrishnan R and Satija I I 2004 *Phys. Lett.* at press (*Preprint* nlin.CD/0303071)
Berry M V 1984 *Proc. R. Soc. Lond.* A **392** 45
Brennen G, Pupillo G, Rey A M, Clark C W and Williams C J 2003 *Preprint* quant-ph/0312069
Boyland P 1994 *Preprint* math.DS/9408216 v1
Davis K B *et al* 1995 *Phys. Rev. Lett.* **75** 3969
Du J *et al* 2002 *Preprint* quant-ph/0207022
Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series and Products* (New York: Academic)
Hannay J H 1985 *J. Phys. A: Math. Gen.* **18** 221
Jones J A, Vedral V, Ekert A and Castagnoli G 2000 *Nature* **403** 869
Leboeuf P, Kurchan J, Feingold M and Arovas D P 1990 *Phys. Rev. Lett.* **65** 3076
Mukunda N and Simon R 1993 *Ann. Phys., NY* **228** 205
Nogamin Y 1991 *Am. J. Phys.* **59** 64
Rapp D 1970 *Quantum Mechanics* (New York: Holt, Rinehart and Winston)
Robbins J M and Berry M V 1992 *Proc. R. Soc. Lond.* A **436** 631
Samuel J and Bhandari R 1988 *Phys. Rev. Lett.* **60** 2339
Satija I I and Balakrishnan R 2004 *Proc. Kyoto Conf. on Quantum Mechanics and Chaos: From Fundamental Problems Through Nanosciences* p 782 (*Preprint* nlin.CD/0403008)
Seshadri S, Lakshmibala S and Balakrishnan V 1999 *Phys. Lett.* A **256** 15
Shapere A and Wilczek F 1989 *Geometric Phases in Physics* (Singapore: World Scientific)
Shaw S W and Holmes P J 1983 *Phys. Rev. Lett.* **51** 623
Song D Y 1999 *Phys. Rev.* A **59** 2616
Spanier J and Oldham K B 1987 *Atlas of Functions* (New York: Hemisphere) p 221
Thompson J M T and Stewart H B 1986 *Nonlinear Dynamics and Chaos* (New York: Wiley) chapter 14, and references therein
Zanardi P and Rasetti M 1999 *Phys. Lett.* A **264** 94